Poles and Transmission Zeros of Flexible Spacecraft Control Systems

Takashi Kida* and Yoshiaki Ohkami†
National Aerospace Laboratory, Tokyo, Japan
and
Shigeo Sambongi‡
National Space Development Agency, Tokyo, Japan

A method is presented for numerically determining multivariable poles and zeros of nonspinning flexible space structures consisting of one rigid part and elastic appendages. Among many definitions of system zeros, transmission zeros are investigated relating the force/torque input applied at an arbitrary point and the translational/angular displacement output at another point. Computation algorithms are obtained for the cases when the modal data are given in terms of constrained modes and unconstrained modes. Investigations on duality and equivalency of these two approaches are made, and the relations of the numerical structures of the algorithms to controllability and observability conditions are also clarified. The proposed algorithms are demonstrated for a simple numerical model.

Introduction

IN the LSS attitude and figure/shape control problems, large dimensional multivariable control is required and major efforts for analysis and synthesis have been devoted to the time-domain approach. Actually, analytical and numerical benefits for control system analysis and systematic methods for controller design purposes are provided. However, it is frequently pointed out that these approaches may result in the fragile controllers to the modeling errors and that fatal performance degradation such as spillover phenomena can occur in the worst cases.

In view of this, the frequency-domain approach is expected to give another perspective to the LSS modeling and control problems in cooperation with the time-domain approaches in a complementary manner. In the control system engineering, the frequency-domain approach has been revalued in the past few years and it has turned out to play an important role in investigating the robustness of the multivariable control system with the modeling uncertainties even if the control system is designed by modern control theories. The investigations in the frequency-domain are also useful for designing the pole-zero assignment regulators or the decoupling controllers.

This paper discusses a method of the frequency-domain representation of a typical LSS model in terms of multivariable poles and zeros, intending to investigate the dynamical structure. Based on an early work[1] which treated only a limited case, determination algorithms are extended to include more general input-output relations for a class of flexible space structures represented in terms of constrained and unconstrained modes.[2] Among many definitions of multivariable zeros,[3-9] the "transmission zeros" are employed. Although many determination algorithms have been studied for general linear systems, they usually require complicated transformations or indirect numerical calculations. By specifying the zero determination problems to the LSS control systems, the proposed algorithm gives a straightforward procedure requiring only an eigenvalue solution with some matrix manipulations in the same manner as pole determinations.

The obtained algorithm is investigated and it is clarified that controllability and observability conditions are closely related to matrix nonsingularity conditions required numerically. Moreover, it is proved that constrained and unconstrained modal frequencies are in a dual situation from the viewpoint of multivariable poles and zeros. In other words, modal frequencies obtained by constrained modal analysis are transmission zeros of the LSS system and equal to the results of the proposed algorithm for the unconstrained mode model. On the other hand, unconstrained modal analysis gives system poles directly and equivalent to the eigenvalues solved for the constrained mode model by the algorithm under some conditions.

System Description

Discussions in this paper are restricted to a class of flexible spacecraft consisting of N_a elastic appendages attached to a central rigid body or a reference body as shown in Fig. 1. Under the assumptions that the system is nongyroscopic and elastic deformations are small, the linearized dynamical equations are given by

$$M^*\ddot{q} + p^T\ddot{w} = f_v \qquad \text{(Total vehicle motion)} \tag{1a}$$

$$M\ddot{w} + D\dot{w} + Kw + p\ddot{q} = f_e \qquad \text{(Appendage vibration)} \tag{1b}$$

where $q^T = [r^T, \theta^T] \in R^6$ consists of the mass center translational displacement $r \in R^3$ and the rotational angle $\theta \in R^3$ of the total vehicle, and the variable $w^T = [w_1^T, \ldots w_{Na}^T]$ represents the small deformation w_i of the ith appendage. The total mass matrix $M^* = \text{block-diag}[m^*; I^*]$ combines the total vehicle mass m^* and its inertia matrix I^* referred to the mass center in the undeformed state. The matrices M, K, and D are also block diagonal. The diagonal block entries of the matrices M and K are the mass matrices M_i and the stiffness matrices K_i $(i=1,\ldots,Na)$ of the ith appendage, and both are intrinsically symmetric and positive definite, $M_i = M_i^T > 0$, $K_i = K_i^T > 0$. Additionally, the damping coefficient matrices D_i $(i=1,\ldots,Na)$ are assumed positive definite and symmetric $D_i = D_i^T > 0$. The matrix p denotes the dynamical coupling

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*Researcher, Space Technology Research Group.
†Senior Researcher, Space Technology Research Group. Member AIAA.
‡Chief Development Engineer, First Satellite Design Group.

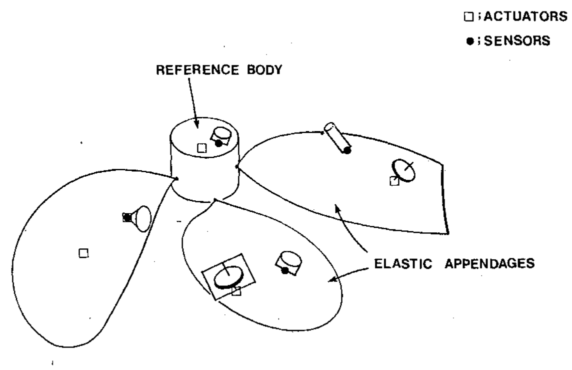

Fig. 1 Flexible spacecraft modeling.

between the elastic vibration of the appendages and the vehicle translational and rotational motions. Consider the case when actuators and sensors are distributed on both the central rigid body and the elastic appendages. Inputs to the system then can be defined by the following linear equations

$$f_v = u_o + B_{ve} u_e \tag{2a}$$

$$f_e = B_{ee} u_e \tag{2b}$$

$$u_e = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix} \tag{2c}$$

where u_o is the control force and torque due to the actuators located on the reference body and u_e at the k actuator points located on the appendages. Similarly, output of the system is assumed to measure the translational and angular displacement of the total vehicle y_o and the elastic deformation y_e at m sensor points located on the appendages.

$$y_o = q \tag{3a}$$

$$y_e = C_{eo} q + C_{ee} w \tag{3b}$$

$$y_e = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} \tag{3c}$$

It is noted that the total vehicle motion of Eq. (1a) is excited by the sum of u_o and u_e, while the appendage vibration is excited only by u_e. The observation equations (Eqs. (3)) have a duality in the sense that the measurements of appendage sensors are affected by the sum of q and w, while that of primary body sensors are only by q. The meanings of the influence matrices B_{ve}, B_{ee} and the observation matrices C_{eo} and C_{ee} are inspected in what immediately follows.

It is natural to consider that various types of actuators may be used to control complex flexible spacecrafts, and they are reflected in the dimensions and the structures of the input vectors u_o and u_e and the influence matrices. This is the case if we consider the practical requirements of redundancy or some operational constraints and so on. The influence matrices combine and distribute the elements of the input vectors, according to actuator locations, force directions and types of actuators, resulting in the force and torque of the directions assigned to the variables q or w. On the other hand, the given system responds to the resultant force/torque input mentioned above, independent of the types and number of implemented actuators. Moreover, this resulting input characterizes the essential structures of the flexible spacecraft control systems, rather than the actual mounting configurations of actuators. With this consideration, net forces and torques are here treated, leaving actuator steering law untouched. The observation equations are in a similar situation, and translational and angular displacements are treated here leaving sensor characteristics untouched, either.

The multi-input multi-output (MIMO) system with the inputs, Eq. (2), and the outputs, Eq. (3), is described in terms of matrix transfer functions G_{oo}, G_{oe}, G_{eo}, and G_{ee} by

$$\begin{bmatrix} y_o \\ y_e \end{bmatrix} = \begin{bmatrix} G_{oo}(s) & G_{oe}(s) \\ G_{eo}(s) & G_{ee}(s) \end{bmatrix} \begin{bmatrix} u_o \\ u_e \end{bmatrix} \tag{4}$$

Considering the structures of u_e and y_e mentioned above, the matrices G_{oo}, G_{eo}, G_{oe} and G_{ee} are further broken down into 6×6 dimensional submatrices. Equation (4) is rewritten by linear combinations of the following four types of input-output relations

$$y_o = G_{oo}(s) u_o \tag{5a}$$

$$y_o = G_{oj}(s) u_j \tag{5b}$$

$$y_i = G_{io}(s) u_o \tag{5c}$$

$$y_i = G_{ij}(s) u_j \tag{5d}$$

Eqs. (5a-d) are interpreted as the frequency-domain response from resultant force/torque inputs at a certain point j, either on the rigid body or the appendage of the vehicle, to translation/rotation measurements at another point i. It is noted that all of Eq. (5) denotes 6 input-6 output multivariable systems with 6×6 square matrix transfer functions.

The primitive model of Eq. (1) can be efficiently represented in the modal space. Hughes[2] has categorized the different approaches into constrained modes and unconstrained modes. Constrained modes are calculated for each appendage by fixing the reference rigid body by $q = 0$ in Eq. (1). Using the obtained mode shape ϕ_i^c of the ith appendage and transforming w_i into η_i by $w_i = \phi_i^c \eta_i$, Eqs. (1)-(3) become

$$M^* \ddot{q} + P^T \ddot{\eta} = u_o + B_{ve} u_e \tag{6a}$$

$$\ddot{\eta} + D^c \dot{\eta} + \omega^2 \eta + P \ddot{q} = \phi^{cT} B_{ee} u_e \tag{6b}$$

$$y_o = q \tag{6c}$$

$$y_e = C_{eo} q + C_{ee} \phi^c \eta \tag{6d}$$

where ϕ^c, D^c, and ω^2 are block diagonal matrices whose diagonal entries are ϕ_i^c, D_i^c, ω_i^2, respectively. The modal coordinates denoted by $\eta^T = [\eta_1^T, \ldots \eta_{Na}^T]$ collect the constrained modes of all appendages. The retained mode number is assumed N where $N = N_1 + \cdots + N_{Na}$. The coupling coefficient matrix $P^T = p^T \phi^c$ includes linear modal momentum coefficients and angular modal momentum coefficients.

Alternatively, Eqs. (1)-(3) are also represented by unconstrained modes as

$$\ddot{\eta}_o = \phi_1^{uT} u_o + \phi_1^{uT} B_{ve} u_e \tag{7a}$$

$$\ddot{\eta}_e + D^u \dot{\eta}_e + \sigma^2 \eta_e = \phi_2^{uT} u_o + \left(\phi_2^{uT} B_{ve} + \phi_3^{uT} B_{ee} \right) u_e \tag{7b}$$

$$y_o = \phi_1^u \eta_o + \phi_2^u \eta_e \tag{7c}$$

$$y_e = C_{eo} \phi_1^u \eta_o + \left(C_{eo} \phi_2^u + C_{ee} \phi_3^u \right) \eta_e \tag{7d}$$

In Eq. (7), unconstrained modal coordinates η and corresponding eigenvector ϕ^u are defined by

$$\eta = \begin{bmatrix} \eta_o \\ \eta_e \end{bmatrix} \tag{8}$$

$$\phi^u = \begin{bmatrix} \phi_1^u & \phi_2^u \\ 0 & \phi_3^u \end{bmatrix} \tag{9}$$

where η_o and η_e are rigid body modes and elastic vibration modes, respectively, and ϕ_1^u, ϕ_2^u and ϕ_3^u are the corresponding mode shapes. These unconstrained modal parameters are obtained either by direct modal analysis of Eq. (1) under the free-free boundary condition or by the transformation from Eq. (6) of the constrained modal space representation.

Determination of Poles and Transmission Zeros

Although the definition of multivariable poles is unique and well known, various definitions of multivariable zeros have been proposed,[3-9] some of which are overlapping in some appropriate conditions.[5] One of the representative definitions of transmission zeros is given by Davison and Wang[3,4] for a general multivariable linear system,

$$\dot{x} = Ax + Bu;\ x \in R^n,\ u \in R^m,\ y \in R^r$$
$$y = Cx + Du \tag{10a}$$

The transmission zeros of (A, B, C, D) are defined as the collection of all complex numbers λ which satisfy

$$rk\begin{bmatrix} A - \lambda U & B \\ C & D \end{bmatrix} < n + \min(r, m) \tag{10b}$$

where $rk[.]$ hereafter denotes the matrix rank. Laub and Moore[6] also employed an equivalent definition of the Davison and Wang set of transmission zeros and showed that these are equivalent to the MacFarlane and Karcanias invariant zeros[5] under the assumption that the system is nondegenerate. The Smith zeros of Rosenbrock's system matrix are commonly called invariant zeros, which do not coincide with transmission zeros generally. MacFarlane and Karcanias also discussed the relation between the invariant zeros and their transmission zeros, defined as zeros of the transfer matrix. The invariant zeros include these transmission zeros and some of the decoupling zeros. However, in a completely controllable/observable case, the sets of invariant zeros and transmission zeros are the same. The general situation when the given system is not controllable and observable is quite complicated.[5] In a completely controllable/observable case, it is shown[4] that transmission zeros given by Eq. (10) are also equal to the zeros of the polynomial obtained by multiplying all numerator polynomials of the Smith-McMillan form of the transfer function $G(s) = C(sU - A)^{-1}B + D$. Therefore, it is summarized that some of the transmission zeros defined in the various ways are equivalent to each other in case of nondegenerate, completely controllable, and observable systems. Other definitions of multivariable zeros and transmission zeros are described in details by Francis and Wohnam.[8]

On returning to the underlying typical type of flexible spacecraft control systems, Eqs. (6) and (7), with resultant inputs and outputs, it turns out that they have the characteristics of nondegeneracy and $r = m$ with $D = 0$. Under the assumption that sensors and actuators are located so as to satisfy controllability/observability conditions, the transmission zeros by Eq. (10) of the flexible spacecraft control systems are simplified to be equivalent to Kwakernaak and Sivan's transmission zeros[9] by replacing rank conditions in Eq. (10b) with matrix determinants.

According to Kwakernaak and Sivan, zeros of the multivariable system

$$y(s) = G(s)u(s) \tag{11a}$$

are zeros of the polynomial $n(s)$ defined by

$$n(s) = d(s)\det[G(s)] \tag{11b}$$

where $G(s) = C(sU - A)^{-1}B$ is the matrix transfer function and $d(s) = \det[sU - A]$ is the characteristic polynomial of the system. The situation when the controllability/observability condition is violated is discussed later in this paper.

Numerous algorithms[3,5-7] have been developed recently to compute transmission zeros of general MIMO systems of Eq. (10a) with or without some constraints. Emami-Naeini and Dooren[7] have clearly classified these existing algorithms into three kinds of techniques, which follow, and evaluated their advantages and disadvantages in terms of numerical aspects and structural simplicity. Davison and Wang's method[3] utilizes the invariance property of transmission zeros under high-gain output feedback to determine their locations. This basically simple algorithm requires only the solving of eigenvalue problems of the closed-loop system matrix. It has, however, the disadvantages of a possibility of numerical ill conditioning and the necessity of sorting true zeros from the obtained results. The second method (representatively due to Laub and Moore[6]) solves generalized eigenvalue problems of the system matrix by the QZ algorithm. This has numerical stability properties. However, sorting is still required and squaring must be performed for general systems. The last class of techniques is based on the Kronecker canonical form of the system matrix which requires two steps, a unitary transformation of the system matrix to the upper Schur form, and the QZ algorithm solution. These general purpose algorithms have been developed intending to solve a wider class of zeros of general systems, including degenerate, nonsquare, uncontrollable-unobservable cases.

The algorithm discussed in this paper is, in turn, planned to treat a specified class of flexible spacecraft mentioned above and utilizes the specific structure of the systems described by second-order-matrix differential equations of Eqs. (6) and (7). Because of this fact, this method has the advantages of its basic simplicity and numerical stability without any pre- or post-manipulations. The required calculation is needed only to solve the eigenvalues in the ordinary sense of a numerically well-conditioned real matrix. This eigenvalue computation routine is totally equivalent to the pole calculation procedure. Therefore, the algorithm provides a means to calculate poles and transmission zeros in a unified manner simply by constructing a matrix with eigenvalues transmission zeros or poles.

The following algorithms are the main results of this paper. Table 1 summarizes the algorithm to determine poles and transmission zeros from jth input to ith output where $j = 0, 1, \ldots, k$ and $i = 0, 1, \ldots, m$. The matrices denoted by $A^c, A^u, \ldots$ are of the forms which follow.

Algorithm I: Eigenvalue Problem of Constrained Mode Model

If the modal data is given in terms of "constrained mode," poles and transmission zeros are the eigenvalues of the matrix of the form

$$A = \begin{bmatrix} 0 & U_N \\ -M\omega^2 & -MD^c \end{bmatrix} \tag{12a}$$

where ω^2 and D^c are constrained modal stiffness and damping matrices, respectively. According to the objectives, the content of M should be replaced by

$$\text{For } A^c:\quad M = \left(U_N - P(M^*)^{-1}P^T\right)^{-1} \tag{12b}$$

$$\text{For } A^{c1}:\quad M = \left(U_N - P(C_{eo})^{-1}C_{ee}\phi^c\right)^{-1} \tag{12c}$$

$$\text{For } A^{c2}:\quad M = \left(U_N - P(B_{ve}^T)^{-1}B_{ee}^T\phi^c\right)^{-1} \tag{12d}$$

where the eigenvalue calculations of A^c, A^{c1} and A^{c2} provides poles and transmission zeros of G_{io} and G_{oj}, respectively.

Proof

The procedure for obtaining transmission zeros of $G_{oo}(s)$ is shown as a representative example. The matrix transfer function $G_{oo}(s)$ is written in the explicit form from Eq. (6) under the assumption all initial values are zero, i.e.,

$$G_{oo}(s)=s^{-2}\left[M^{*}-P^{T}Q(s)P\right]^{-1},$$

$$Q(s)=s^{2}\left[s^{2}U_{N}+sD^{c}+\omega^{2}\right]^{-1}$$

Transmission zeros of G_{oo} are, as defined by Eq. (11b), zeros of $n_{oo}(s)=d(s)\det[G_{oo}(s)]$, where $d(s)$ is the characteristic polynomial determined naturally independent from the input-output relations. It is not hard to see that $d(s)=s^{12}\det[sU_{2N}-A^{c}]$, where A^{c} has the form as described in Eq. (12b). On the term of $\det[G_{oo}(s)]$, the following reduction can be made, if matrices M^{*} and $[U_{N}-PM^{*-1}P^{T}]$ are nonsingular

$$\det\left[G_{oo}(s)\right]=s^{-12}/\det\left[M^{*}-P^{T}Q(s)P\right]$$

$$\det\left[M^{*}-P^{T}Q(s)P\right]$$

$$=\det M^{*}\det\left[sU_{2N}-A^{c}\right]/\det M^{c}\det\left[s^{2}U_{N}+sD^{c}+\omega^{2}\right]$$

where $M^{c}=[U_{N}-PM^{*-1}P^{T}]^{-1}$. Therefore $n_{oo}(s)=\det[s^{2}U_{N}+sD^{c}+\omega^{2}]\det M^{c}/\det M^{*}$ implying that transmission zeros of G_{oo} are zeros of the polynomial $\det[s^{2}U_{N}+sD^{c}+\omega^{2}]$. It can be observed that no transmission zero exists at the origin and the number of transmission zeros is equal to twice of the elastic modes number, including the possibility of pole-zero cancellation. The values of transmission zeros are proved equal to the constrained modal frequencies. In the undamped special case $D^{c}=0$, they are $\pm j\omega_{1},\ldots,\pm j\omega_{N}$. Similar proofs can be made and omitted here on the remaining matrix transfer functions G_{io} and G_{oj}.

Algorithm II: Eigenvalue Problem of Unconstrained Mode Model

If the modal data is given in terms of "unconstrained mode," poles correspond to the modal frequencies with modal damping D^{u}, and transmission zeros are eigenvalues of the matrix of the form

$$A=\begin{bmatrix}0 & U_{N}\\ -M\sigma^{2} & -MD^{u}\end{bmatrix}\tag{13a}$$

where σ^{2} and D^{u} are unconstrained stiffness and damping matrices, and M should be read as

For A^{u}:

$$M=\left(U_{N}+\phi_{2}^{uT}\left(\phi_{1}^{u}\phi_{1}^{uT}\right)^{-1}\phi_{2}^{u}\right)^{-1}\tag{13b}$$

For A^{u1}:

$$M=\left(U_{N}+\phi_{2}^{uT}\left(C_{eo}\phi_{1}^{u}\phi_{1}^{uT}\right)^{-1}\left(C_{eo}\phi_{2}^{u}+C_{ee}\phi_{3}^{u}\right)\right)^{-1}\tag{13c}$$

For A^{u2}:

$$M=\left(U_{N}+\left(\phi_{2}^{uT}B_{ve}+\phi_{3}^{uT}B_{ee}\right)\left(\phi_{1}^{u}\phi_{1}^{uT}B_{ve}\right)^{-1}\phi_{2}^{u}\right)^{-1}\tag{13d}$$

For A^{u3}:

$$M=\Big(U_{N}+\left(\phi_{2}^{uT}B_{ve}+\phi_{3}^{uT}B_{ee}\right)\times\left(C_{eo}\phi_{1}^{u}\phi_{1}^{uT}B_{ve}\right)^{-1}\left(C_{eo}\phi_{2}^{u}+C_{ee}\phi_{3}^{u}\right)\Big)^{-1}\tag{13e}$$

according to the objectives, i.e., the eigenvalues of A^{u}, A^{u1}, A^{u2}, and A^{u3} are transmission zeros of G_{oo}, G_{io}, G_{oj}, and G_{ij}, respectively.

Proof

Poles and transmission zeros of only G_{oo} are detailed again. The explicit form of the transfer function is $G_{oo}(s)=s^{-2}[\phi_{1}^{u}\phi_{1}^{uT}+\phi_{2}^{u}R(s)\phi_{2}^{uT}]$ from Eq. (7), where $R(s)=s^{2}[s^{2}U_{N}+sD^{u}+\sigma^{2}]^{-1}$. Then transmission zeros are zeros of the numerator polynomial of $G_{oo}(s)$, $n_{oo}(s)=d(s)s^{-12}\det[\phi_{1}^{u}\phi_{1}^{uT}+\phi_{2}^{u}R(s)\phi_{2}^{uT}]$. The denominator polynomial $d(s)$ is the characteristic equation of Eq. (7)

$$d(s)=s^{12}\det[sU_{2N}-F],\ F=\begin{bmatrix}0 & U_{N}\\ -\sigma^{2} & -D^{u}\end{bmatrix}$$

where F is the partitioned block of the system matrix of Eqs. (7) corresponding to the unconstrained vibration mode, and $d(s)$ is reduced to $d(s)=s^{12}\det[s^{2}U_{N}+sD^{u}+\sigma^{2}]$. This fact means poles are the unconstrained modal frequencies having duplicated poles at the origin. Especially for the undamped case, poles are twelve zeros and $\pm j\sigma_{1},\ldots,\pm j\sigma_{N}$. By utilizing the relation

$$\det\left[\phi_{1}^{u}\phi_{1}^{uT}+\phi_{2}^{u}R(s)\phi_{2}^{uT}\right]$$

$$=\det\left[\phi_{1}^{u}\phi_{1}^{uT}\right]\det\left[sU_{2N}-A^{u}\right]/\det M^{u}\det\left[s^{2}U_{N}+sD^{u}+\sigma^{2}\right]$$

based on the assumption $\phi_{1}^{u}\phi_{1}^{uT}$ and $[U_{N}+\phi_{2}^{uT}(\phi_{1}^{u}\phi_{1}^{uT})^{-1}\phi_{2}^{u}]$ are nonsingular, where A^{u} and M^{u} are shown as in Eqs. (13a) and (13b). Transmission zeros are zeros of the numerator polynomial $n_{oo}=\det[sU_{2N}-A^{u}]\det[\phi_{1}^{u}\phi_{1}^{Tu}]/\det M^{u}$. Equations (13a) and (13b) of the proposed algorithm are thus proved. Transmission zeros of G_{io}, G_{oj}, and G_{ij} are obtained in the similar manner, and their proofs are omitted here.

Investigations on Duality and Nonsingularity

Duality and Equivalency

Focusing attention on the transfer function G_{oo}, where both sensors and actuators are located on the reference rigid body, the two modeling approaches of "constrained mode" and "unconstrained mode" are dual in the sense that the constrained modal frequencies ω_{i} $(i=1,\ldots,N)$ are transmission zeros of the system, Eqs. (1)-(3), while the unconstrained modal frequencies σ_{i} $(i=1,\ldots,N)$ are poles of the system, where N is the retained mode number. The duality is proved in the above chapter and obviously seen from Table 1. In what follows, it will be shown that poles and transmission zeros computed from constrained mode are equivalent to those from unconstrained mode. That is

⟨transmission zeros⟩

↔ ⟨modal frequencies of constrained mode with modal damping⟩

↔ ⟨eigenvalues of A^{u}, Eq. (13b)⟩ (Equivalency relation A)

⟨poles⟩

↔ ⟨modal frequencies of unconstrained mode with modal damping⟩

↔ ⟨eigenvalues of A^{c}, Eq. (12b)⟩ (Equivalency relation B)

For convenience of the detailed discussions, the problem is limited to the case without any significant loss of generality that 1) the appendages vibration mode is antisymmetric and has no significant influence on the vehicle translational motion, 2) the number of retained modes N is sufficiently large for modeling the original motion without causing any truncation error, and 3) N is assumed common to the two modal analysis approaches.

Table 1 Algorithm summary

	Modal Data	
	Constrained mode (Reference body fixed)	Unconstrained mode
Poles of $G_{ij}(s)$	Eigenvalues of A^c	Modal frequencies
Transmission zeros of $G_{oo}(s)$	Modal frequencies	Eigenvalues of A^u
Transmission zeros of $G_{jo}(s)$	Eigenvalues of A^{c1}	Eigenvalues of A^{u1}
Transmission zeros of $G_{oj}(s)$	Eigenvalues of A^{c2}	Eigenvalues of A^{u2}
Transmission zeros of $G_{ij}(s)$	Not available	Eigenvalues of A^{u3}

Based on these assumptions, constrained mode shape ϕ^c and unconstrained mode shape ϕ^u have the relations

$$\phi_1^u = (I^*)^{-1/2},\ \phi_2^u = k,\ \phi_3^u = \phi^c t,\ P^T t + I^* k = 0 \quad (14)$$

It is noted that t is the $N \times N$ square matrix, since ϕ_3^u and ϕ^c have the same dimension as mentioned above. From the matrices defined in Eq. (14) and the mode shape definitions, the following modal identities hold[2]

$$t^T \omega^2 t = \sigma^2,\ t^T D^c t = D^u,\ t^T t - k^T I^* k = U_N \quad (15)$$

Based on these relations, the eigenvalues of A^u are given by

$$\begin{aligned}\det[sU_{2N} - A^u] &= \det M^u \det\left[s^2 (M^u)^{-1} + sD^u + \sigma^2\right] \\ &= \det M^u \det\left[s^2\left(U_N + \phi_2^u\left(\phi_1^u \phi_1^{uT}\right)^{-1} \phi_2^u\right) + sD^u + \sigma^2\right] \\ &= \det M^u \det\left[s^2\left(U_N + k^T I^* k\right) + s t^T D^c t + t^T \omega^2 t\right] \\ &= \det M^u \det\left[t^T\left(s^2 U_N + sD^c + \omega^2\right) t\right]\end{aligned}$$

and equal to the constrained modal frequencies with modal damping D^c.

Secondarily, the poles equivalency is proved by reducing the eigenvalues of A^c from Eq. (12b) to

$$\begin{aligned}&\det[sU_{2N} - A^c] \\ &\quad = \det M^u \det\left[s^2\left(U_N - P(I^*)^{-1} P^T\right) + sD^c + \omega^2\right] = 0\end{aligned}$$

Pre- and post-multiplying the above equation by t^T and t, respectively, yields

$$\begin{aligned}&\det\left[t^T (sU_{2N} - A^c) t\right] \\ &\quad = \det M^u \det\left[s^2\left(t^T t - t^T P (I^*)^{-1} P^T t\right) + s t^T D^c t + t^T \omega^2 t\right] \\ &\quad = \det M^u \det\left[s^2\left(t^T t - k^T I^* k\right) + s t^T D^c t + t^T \omega^2 t\right] \\ &\quad = \det M^u \det\left[s^2 U_N + sD^u + \sigma^2\right] \\ &\quad = 0\end{aligned}$$

indicating that the eigenvalues of A^c are equal to the unconstrained modal frequencies with modal damping D^u. It is noted that equivalence relations are violated when significant modal truncation error exists. Although treatments of such cases are beyond the scope of this paper, it is pointed out that the poles and transmission zeros equivalency is an interpretation of the modal identities in the MIMO frequency representation. This serves as a modal truncation criterion.

Nonsingularity Conditions of Eigenvalue Problems

In constructing matrix A to yield poles and transmission zeros as in Eqs. (12) and (13), some nonsingularity conditions must be satisfied as described in the proofs of algorithms. For the unconstrained mode approach, Eq. (13), those nonsingularity conditions are equivalent to the following rank conditions.

For A^c:

$$rk\left[\phi_1^u \phi_1^{uT}\right] = 6 \quad (16a)$$

$$rk\left[\phi_1^u \phi_1^{uT} + \phi_2^u \phi_2^{uT}\right] = 6 \quad (16b)$$

For A^{u1}:

$$rk\left[C_{eo} \phi_1^u \phi_1^{uT}\right] = 6 \quad (17a)$$

$$rk\left[C_{eo} \phi_1^u \phi_1^{uT} + \left(C_{eo} \phi_2^u + C_{ee} \phi_3^u\right) \phi_2^{uT}\right] = 6 \quad (17b)$$

For A^{u2}:

$$rk\left[\phi_1^u \phi_1^{uT} B_{ve}\right] = 6 \quad (18a)$$

$$rk\left[\phi_1^u \phi_1^{uT} B_{ve} + \phi_2^u\left(\phi_2^{uT} B_{ve} + \phi_3^{uT} B_{ee}\right)\right] = 6 \quad (18b)$$

For A^{u3}:

$$rk\left[C_{eo} \phi_1^u \phi_1^{uT} B_{ve}\right] = 6 \quad (19a)$$

$$\begin{aligned}&rk\left[C_{eo} \phi_1^u \phi_1^{uT} B_{ve} + \left(C_{eo} \phi_2^u + C_{ee} \phi_3^u\right)\right. \\ &\quad \left.\times\left(\phi_2^{uT} B_{ve} + \phi_3^{uT} B_{ee}\right)\right] = 6\end{aligned} \quad (19b)$$

Those conditions can be interpreted from the viewpoint of controllability and observability conditions. For simplicity, only the vehicle attitude motion coupled with the appendage vibration is considered. It is seen that Eq. (16) is directly verified by using modal identities (14) again and the identity of the form

$$kk^T = I_r^{-1} - (I^*)^{-1} \quad (20)$$

for the complete set of modes,[2] where I_r is the rigid part contribution to I^*. Then the matrices $\phi_1^u \phi_1^{uT} = (I^*)^{-1}$ and $\phi_1^u \phi_1^{uT} + \phi_2^u \phi_2^{uT} = I_r^{-1}$ are nonsingular.

Moreover, it is noted that Eqs. (17a), (18a), and (19a) are a special form of controllability and observability conditions proposed by Hughes and Skelton.[10] Equation (17a), especially, is equivalent to the observability condition and Eq. (18a) to the controllability condition. In addition, Eq. (19a) is valid if and only if the rigid mode is controllable and observable. The conditions of Eqs. (17b), (18b), and (19b) are, in turn, related to controllability/observability conditions of the rigid and the elastic modes. For example, Eq. (19b) reduces to

$$\begin{aligned}&\det\left[C_{eo} \phi_1^u \phi_1^{uT} B_{ve} + \left(C_{eo} \phi_2^u + C_{ee} \phi_3^u\right)\left(\phi_2^{uT} B_{ve} + \phi_3^{uT} B_{ee}\right)\right. \\ &\quad = \det\left\{\left[C_{eo} \phi_1^u;\ C_{eo} \phi_2^u + C_{ee} \phi_3^u\right] \begin{bmatrix} \phi_1^{uT} B_{ve} \\ \phi_2^{uT} B_{ve} + \phi_3^{uT} B_{ee} \end{bmatrix}\right\} \\ &\quad \neq 0\end{aligned}$$

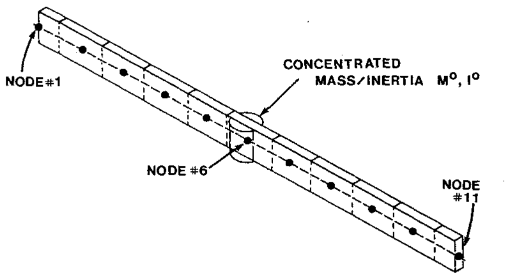

Fig. 2 Finite element beam model.

Table 2 Comparison of transmission zeros of G_{oo}

Mode		1st	2nd	3rd	4th
Constrained modal frequency $\omega_i[Hz]$		4.343	4.343	27.229	27.229
Transmission zeros $[Hz]$	$\mu = 10$	4.343	4.343	27.230	27.230
	$\mu = 1$	4.343	4.345	27.230	27.255
	$\mu = 0.1$	4.347	4.351	27.244	27.591
	$\mu = 0.01$	4.351	4.514	27.886	28.401
	$\mu = 0$	4.352	5.152	27.940	34.368

The necessary condition of this equation is that

$$rk[C_{eo}\phi_1^u;\ C_{eo}\phi_2^u + C_{ee}\phi_3^u] = rk[B_{ve}^T\phi_1^u;\ B_{ve}^T\phi_2^u + B_{ee}^T\phi_3^u] = 3$$

It is noted that the first partitions, $C_{eo}\phi_1^u$ and $B_{ve}^T\phi_1^u$ are observability and controllability matrices of the rigid mode, respectively, and the second partitions are those of the elastic modes.

Numerical Examples

The proposed algorithms are demonstrated for a simple numerical model described in terms of unconstrained modes with emphasis on the transmission zero determination in the case of sensor/actuator located on the primary rigid body. The demonstration model is assumed undamped and nongyroscopic, and consists of one central rigid body and two elastic beams arranged in a symmetric configuration as illustrated in Fig. 2.

The modal analysis is performed numerically by utilizing finite element method (FEM). For this purpose, the elastic part is partitioned into ten finite elements and the rigid part is expressed by a concentrated mass/inertia placed at the center node (node 6), to which the mass center of the total spacecraft corresponds. Then the "constrained modes" of Eq. (6) are obtained by constraining six degrees of freedom of the sixth node for FEM calculation, and the "unconstrained modes" of Eq. (7) are obtained by setting all of the nodes translationally and rotationally free. In the following, only the lowest four modes are considered, i.e., the symmetric and antisymmetric vibration modes attributed to the first and second bending of each appendage. The truncation procedure is based on the fact that the structure is simple, and the lowest two modes are dominant from the viewpoint of modal completeness index.

The strategy of algorithm accuracy evaluation is based on the equivalency relation A. According to the FEM terminology, transmission zeros of G_{oo} are equivalent to the modal frequencies under the condition of node 6 constrained. Algorithm (13), on the other hand, utilizes the modal frequencies and the mode shapes solved for the case of all nodes being free. Relation A implies that both of these results should be identical in the ideal case.

From this viewpoint, the constrained modal frequencies ω_i and transmission zeros z_i obtained from Algorithm II of Eqs. (13a) and (13b) are compared and the numerical results are summarized in Table 2. The parameter μ in the table denotes ratio of the mass matrix elements of the rigid part itself, M_{ij}^o, and the corresponding appendage, M_{ij}^a referred to the total vehicle mass center, i.e., $\mu = M_{ij}^o/M_{ij}^a$ where $M_{ij}^o + M_{ij}^a = M_{ij}^*$, $i, j = 1, \ldots, 6$, with M_{ij}^* being the elements of M^* in Eq. (1). It is known that the modeling accuracy crucially depends on the value of parameter μ, in either constrained or unconstrained mode representation.[11,12] Indeed, the results in Table 2 endorse this fact. Moreover, in the extreme case of $\mu \to 0$ the 2nd and the 4th constrained modal frequencies are not identical to corresponding transmission zeros any longer (although, this case is beyond the reasoning of the original equation validation, transmission zeros can be calculated at least formally).

Conclusion

Poles and transmission zeros of a typical class of flexible spacecrafts are discussed in terms of constrained and unconstrained modes, with clarification of their dynamical structures in the frequency-domain intended. The duality and equivalency relations of poles and zeros of these two modeling approaches are investigated analytically in an ideal case with the modal truncation error is negligible. The proposed algorithms are numerically demonstrated by a simple beam model represented in terms of unconstrained modes. Although the obtained algorithm requires a matrix inversion of dimension $N \times N$, where N is the retained mode number, the matrix structure assures that the inversion is almost always well conditioned. The condition number depends on the degree of controllability and observability.

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